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Some Extremal Problems for Multivariate Polynomials on Convex Bodies

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In this paper we extend some Chebyshev and Remez-type inequalities for multivariate polynomials.

Consider the set P_n of complex valued polynomials of m real variables and of total degree at most n:

$$P_n := \left\{ \sum_{|k| \leqslant n} c_k x^k : c_k \in \mathbb{C}, x \in \mathbb{R}^m \right\}.$$

As usual, for $x = (x_1, ..., x_m) \in \mathbb{R}^m$ and $k = (k_1, ..., k_m) \in \mathbb{Z}_+^m$ we set $x^k = \prod_{j=1}^m x_j^{k_j}$ and $|k| = k_1 + \cdots + k_m$. Given a compact set $K \subset \mathbb{R}^m$ denote by $\| p \|_{C(K)} := \sup_{x \in K} |p(x)|$ the uniform norm of p on K, and let $\eta_m(K)$ be the m-dimensional Lebesque measure of $K \subset \mathbb{R}^m$.

In this paper we shall consider two basic problems.

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A. *The Chebyshev-Type Problem.* This consists in estimating the norm of a polynomial on a set F provided that its norm on another set K is known, i.e., finding for a given F, $K \subset \mathbb{R}^m$

$$\sup\left\{\frac{\|p\|_{C(F)}}{\|p\|_{C(K)}} : p \in P_n,\ p \not\equiv 0\right\}.$$

B. *The Remez-Type Problem.* This is related to estimating the norm of a polynomial on a set $K \subset \mathbb{R}^m$ by its norm on a subset of K of "large" measure, that is, determining for a given $0 < \varepsilon < 1$

$$\sup\left\{\frac{\|p\|_{C(K)}}{\|p\|_{C(F)}} : p \in P_n,\ p \not\equiv 0;\ F \subset K,\ \eta_m(F) \geqslant (1-\varepsilon)\,\eta_m(K)\right\}.$$

The solutions to the above problems are well-known in the univariate case, when K is a finite interval on the real line. These solutions are based on the Chebyshev polynomial of first kind defined by $T_n(x) = \cos n \arccos x$ $(|x| \leqslant 1)$.

Chebyshev Inequality [1, Sect. 5.1, Eq. 2]. For every $p \in P_n$ $(m=1)$ and $x \in \mathbb{R} \backslash [-1, 1]$ we have

$$|p(x)| \leqslant T_n(|x|)\,\|p\|_{C[-1,1]}. \tag{1}$$

Remez Inequality [8]. For any $p \in P_n$ $(m=1)$ satisfying $\eta_1\{x \in [-1, 1]: |p(x)| \leqslant 1\} \geqslant 2(1-\varepsilon)$ with some $0 < \varepsilon < 1$ we have

$$\|p\|_{C[-1,1]} \leqslant T_n\left(\frac{1+\varepsilon}{1-\varepsilon}\right). \tag{2}$$

Estimates (1) and (2) are clearly sharp, since they are attained for T_n or its translates. Thus (1) and (2) provide exact solutions to Problems A and B above in the case when K is a line segment. (It should be noted that in [8] inequality (2) is verified for polynomials with real coefficients, but the complex version is a straightforward consequence of the real estimate.)

The purpose of this paper is to extend the Chebyshev and Remez inequalities to the multivariate setting. In the first section we shall consider the multivariate Chebyshev problem. In the second part the Remez problem for multivariate polynomials will be studied.

MULTIVARIATE CHEBYSHEV INEQUALITY

We shall need some additional notations and definitions. Let $K \subset \mathbb{R}^m$ be a compact set; $a, b \in K$ and $c \in \mathbb{R}^m$, $c \neq 0$ be such that $\langle c, a\rangle \leqslant \langle c, b\rangle$. (As usual, $\langle x, y\rangle$ denotes the inner product of $x, y \in \mathbb{R}^m$.) Then the set of points

$$S := \{x \in \mathbb{R}^m : \langle c, a\rangle \leqslant \langle c, x\rangle \leqslant \langle c, b\rangle\} \qquad (a, b \in K) \tag{3}$$

is called a *supporting strip* for K if $K \subset S$. Thus a supporting strip of K is the set of points enclosed between two parallel supporting hyperplanes of K. Denote by $S(K)$ the set of all supporting strips of K. Furthermore, for a supporting strip S given by (3), its α-extension $S_\alpha (\alpha > 1)$ is defined by

$$S_\alpha := \left\{ x \in \mathbb{R}^m : \left\langle c, a - (\alpha - 1)\frac{b-a}{2} \right\rangle \leqslant \langle c, x\rangle \leqslant \left\langle c, b + (\alpha - 1)\frac{b-a}{2} \right\rangle \right\}. \tag{4}$$

It should be noted that if the distance between boundary hyperplanes of S is ϱ, then for S_α this distance is $\alpha\varrho$.

This leads to the notion of α-extension of a set $K \subset \mathbb{R}^m$:

$$K_\alpha := \{\cap S_\alpha : S \in S(K)\}, \qquad \alpha > 1.$$

Evidently, $K \subset K_\alpha \subset K_\beta$ $(1 < \alpha < \beta)$. Using this notion we can introduce a certain "distance" from a set F to K by

$$\varrho(F, K) := \inf\{\alpha : K_\alpha \supset F\}. \tag{5}$$

Based on this definition we can solve the Chebyshev-type problem in the case when $K \subset \mathbb{R}^m$ is a *convex body*, i.e., a compact convex subset of $\mathbb{R}^m$ with nonempty interior.

THEOREM 1. *Let $K \subset \mathbb{R}^m$ be an arbitrary convex body, and consider a compact set $F \subset \mathbb{R}^m (m \geqslant 1)$. Then*

$$\sup\left\{\frac{\|p\|_{C(F)}}{\|p\|_{C(K)}} : p \in P_n, p \not\equiv 0\right\} = T_n(\varrho(F, K)). \tag{6}$$

Remark. The special case when the underlying set K is *strictly* convex and the compact set F consists of a *single* point (6) was studied by Rivlin and Shapiro [9]. Our approach, based on the notion of α-extensions introduced above, provides a simple geometric solution in the general case of *arbitrary* compact sets F. Another feature of the proof of Theorem 1 is the fact that it applies to *any* convex body K. The case when $F = B^m$, the unit ball in $\mathbb{R}^m$, was also considered in Brudnyi and Ganzburg [2].

An essential new element of our approach in comparison with [2] is the use in Theorem 1 of the "distance" ϱ which in turn is based on the notation of α-extensions. The α-extensions of convex bodies essentially lead to "level surfaces" on which ϱ is constant. It will be shown below that under the additional assumption that K is symmetric (i.e., $0 \in K$, and $x \in K$ implies $-x \in K$) we have $K_\alpha = \alpha K = \{\alpha x : x \in K\}$ and

$$\varrho(x, K) = \inf \left\{ \alpha : \frac{x}{\alpha} \in K \right\}.$$

Hence in the symmetric case $\varrho(x, K)$ is the Minkowski functional corresponding to K. It should be also noted that, in general, the α-extension of a nonsymmetric convex body K is not "shape preserving" (e.g., for K a simplex, K_α is not a simplex).

In order to prove Theorem 1 we shall need two auxiliary lemmas. The first lemma provides some information on the geometry of K_α.

LEMMA 1. *Let $K \subset \mathbb{R}^m$ be an arbitrary compact set. Then for any supporting strip $S \in S(K)$ given by* (3) *we have* $b + (\alpha - 1)(b - a)/2$, $a - (\alpha - 1)(b - a)/2 \in K_\alpha (\alpha > 1)$.

Proof. In order to verify Lemma 1 we need to show that $a - (\alpha - 1)(b - a)/2$ and $b + (\alpha - 1)(b - a)/2 \in \tilde{S}_\alpha$, where $\tilde{S}$ is any supporting strip of K. Let $\tilde{S} \in S(K)$ be given by

$$\tilde{S} := \{x \in \mathbb{R}^m : \langle \tilde{c}, \tilde{a} \rangle \leqslant \langle \tilde{c}, x \rangle \leqslant \langle \tilde{c}, \tilde{b} \rangle\}; \qquad \tilde{a}, \tilde{b} \in K, \quad \tilde{c} \in \mathbb{R}^m; \tag{7}$$

$$\tilde{S}_\alpha := \left\{ x \in \mathbb{R}^m : \left\langle \tilde{c}, \tilde{a} - (\alpha - 1) \frac{\tilde{b} - \tilde{a}}{2} \right\rangle \leqslant \langle \tilde{c}, x \rangle \right.$$
$$\left. \leqslant \left\langle \tilde{c}, \tilde{b} + (\alpha - 1) \frac{\tilde{b} - \tilde{a}}{2} \right\rangle \right\}. \tag{8}$$

Since $a, b \in K \subset \tilde{S}$ we have by (7)

$$\langle \tilde{c}, \tilde{a} \rangle \leqslant \langle \tilde{c}, a \rangle \leqslant \langle \tilde{c}, \tilde{b} \rangle; \qquad \langle \tilde{c}, \tilde{a} \rangle \leqslant \langle \tilde{c}, b \rangle \leqslant \langle \tilde{c}, \tilde{b} \rangle. \tag{9}$$

Using (9) we obtain

$$\frac{1 - \alpha}{2} \langle \tilde{c}, a - \tilde{a} \rangle \leqslant 0; \qquad \frac{1 + \alpha}{2} \langle \tilde{c}, b - \tilde{b} \rangle \leqslant 0 \qquad (\alpha > 1).$$

Thus applying the last two inequalities yields

$$\left\langle \tilde{c}, b+(\alpha-1)\frac{b-a}{2}\right\rangle = \frac{1-\alpha}{2}\langle \tilde{c}, a\rangle + \frac{1+\alpha}{2}\langle \tilde{c}, b\rangle$$
$$\leqslant \frac{1-\alpha}{2}\langle \tilde{c}, \tilde{a}\rangle + \frac{1+\alpha}{2}\langle \tilde{c}, \tilde{b}\rangle$$
$$= \left\langle \tilde{c}, \tilde{b}+(\alpha-1)\frac{\tilde{b}-\tilde{a}}{2}\right\rangle. \tag{10}$$

Similarly, by (9)

$$\frac{1-\alpha}{2}\langle \tilde{c}, a-\tilde{b}\rangle \geqslant 0; \qquad \frac{1+\alpha}{2}\langle \tilde{c}, b-\tilde{a}\rangle \geqslant 0 \qquad (\alpha > 1),$$

and hence

$$\left\langle \tilde{c}, b+(\alpha-1)\frac{b-a}{2}\right\rangle = \frac{1-\alpha}{2}\langle \tilde{c}, a\rangle + \frac{1+\alpha}{2}\langle \tilde{c}, b\rangle$$
$$\geqslant \frac{1-\alpha}{2}\langle \tilde{c}, \tilde{b}\rangle + \frac{1+\alpha}{2}\langle \tilde{c}, \tilde{a}\rangle$$
$$= \left\langle \tilde{c}, \tilde{a}-(\alpha-1)\frac{\tilde{b}-\tilde{a}}{2}\right\rangle. \tag{11}$$

Recalling the definition (8) of $\tilde{S}_\alpha$, and using (10) and (11) we obtain that $b+(\alpha-1)(b-a)/2 \in \tilde{S}_\alpha$. Similarly it can be shown that $a-(\alpha-1)(b-a)/2 \in \tilde{S}_\alpha$. ∎

Assume now that K is a convex body in $\mathbb{R}^m$ such that $0 \notin K$, and consider the analogue of the Minkowski functional

$$f(x) := \inf\left\{\alpha > 0 : \frac{x}{\alpha} \in K\right\} \tag{12}$$

defined for $x \in \operatorname{cone}(K) := \{ax : x \in K; a > 0\}$. Usually, the Minkowski functional is defined by (12) when $0 \in K$ but $f(x)$ is a convex functional on $\operatorname{cone}(K)$ when $0 \notin K$, as well. Indeed, if $x, y \in \operatorname{cone}(K)$, then $x/f(x)$, $y/f(y) \in K$, and hence by the convexity of K for any $0 \leqslant \Theta \leqslant 1$

$$\frac{\Theta x+(1-\Theta)\,y}{\Theta f(x)+(1-\Theta)\,f(y)} = \frac{x}{f(x)}\cdot\frac{\Theta f(x)}{\Theta f(x)+(1-\Theta)\,f(y)}$$
$$+\frac{y}{f(y)}\cdot\frac{(1-\Theta)\,f(y)}{\Theta f(x)+(1-\Theta)\,f(y)} \in K.$$

Therefore $f(\Theta x+(1-\Theta)\,y)\leqslant \Theta f(x)+(1-\Theta)\,f(y)$, $0\leqslant\Theta\leqslant 1$. Moreover, $0<f(x)<\infty$ ($x\in\operatorname{cone}(K)$), $f(x)\leqslant 1$ for $x\in K$ (with strict inequality in the interior of K) and $f(tx)=tf(x)$ ($t>0$, $x\in\operatorname{cone}(K)$).

Let us consider now the problem of minimizing $f(x)$ over $x\in K$. The properties of $f(x)$ outlined above yield, in particular, that $f(x)$ is a continuous functional on K, and thus it attains its minimum on K. The next lemma gives a necessary and sufficient condition for this minimum.

LEMMA 2. *Let K be a convex body, $x_0\in K$. Then the following statements are equivalent*:

(i) *$f(x)\geqslant f(x_0)$ for every $x\in K$;*

(ii) *there exist parallel supporting hyperplanes for K at x_0 and $x_0/f(x_0)$.*

Proof. (i) $\mapsto$ (ii). Obviously we have $0<f(x_0)<1$. Consider the ray l emanating from the origin and passing through x_0. Evidently, x_0 must be the point of entry of l into K, and $x_0/f(x_0)$ is its point of exit from K. Consider the open convex set $K_1=K_0+x_0-x_0/f(x_0)$, where $K_0:=\operatorname{Int} K$ is the interior of K. Let us verify that $K_1\cap K_0=\varnothing$. Assume that in contrast there exist $x_1, x_2\in K_0$ such that $x_1=x_2+x_0-x_0/f(x_0)$. Then

$$y:=f(x_0)\,x_2=f(x_0)\left(x_1-x_0+\frac{x_0}{f(x_0)}\right)$$
$$=f(x_0)\,x_1+(1-f(x_0))\,x_0\in K_0. \tag{13}$$

(We use here the fact that if $y\in K$ and $z\in K_0$, then $\Theta y+(1-\Theta)\,z\in K_0$ for every $0<\Theta<1$.) Furthermore, since $x_2\in K_0$ it follows that for some $t<1$ we must have $x_2/t\in K$, i.e., $f(x_2)\leqslant t<1$. This and (13) yield

$$f(y)=f(x_0)\,f(x_2)<f(x_0),$$

contradicting the minimality of x_0.

Thus $K_1\cap K_0=\varnothing$. Since both K_1 and K_0 are convex and open it follows (see [5, p. 130]) that they can be separated. Thus for some nonzero $c\in\mathbb{R}^m$ we have

$$\langle x, c\rangle\leqslant\langle y, c\rangle, \qquad x\in K_1,\ y\in K_0.$$

Since the above relations also hold for the closures of K_0 and $K_1=K_0+x_0-x_0/f(x_0)$, we obtain that

$$\left\langle x+x_0-\frac{x_0}{f(x_0)}, c\right\rangle\leqslant\langle y, c\rangle, \qquad x, y\in K. \tag{14}$$

Set $x = x_0/f(x_0) \in K$ in (14). Then

$$\langle x_0, c\rangle \leqslant \langle y, c\rangle, \qquad y \in K,$$

i.e., $L_1 := \{x \in \mathbb{R}^m: \langle x, c\rangle = \langle x_0, c\rangle\}$ is a supporting plane for K at x_0. Moreover, setting $y = x_0$ in (14) yields

$$\langle x, c\rangle \leqslant \left\langle \frac{x_0}{f(x_0)}, c\right\rangle, \qquad x \in K,$$

i.e., $L_2 := \{x \in \mathbb{R}^m: \langle x, c\rangle = \langle x_0/f(x_0), c\rangle\}$ is a supporting plane for K at $x_0/f(x_0) \in K$. Since hyperplanes L_1 and L_2 are parallel, statement (ii) follows.

(ii) $\mapsto$ (i). Assume that for some $c \in \mathbb{R}^m$ we have

$$\langle x_0, c\rangle \leqslant \langle x, c\rangle \leqslant \left\langle \frac{x_0}{f(x_0)}, c\right\rangle, \qquad x \in K, \tag{15}$$

i.e., there exist parallel supporting hyperplanes for K at x_0 and $x_0/f(x_0)$. In particular, (15) yields that

$$\langle x_0, c\rangle \left(\frac{1}{f(x_0)} - 1\right) \geqslant 0. \tag{16}$$

Moreover, recalling that $f(x) \leqslant 1$ on K, and that the convex body K cannot be contained in a hyperplane, we obtain that $f(x_0) < 1$, i.e.,

$$\langle x_0, c\rangle > 0. \tag{17}$$

Consider now an arbitrary $x_1 \in K$. Then $x_1/f(x_1) \in K$, and we have by (15) that

$$\frac{1}{f(x_0)}\langle x_0, c\rangle \geqslant \left\langle \frac{x_1}{f(x_1)}, c\right\rangle = \frac{1}{f(x_1)}\langle x_1, c\rangle \geqslant \frac{1}{f(x_1)}\langle x_0, c\rangle.$$

Using this last inequality and (17) yields $f(x_1) \geqslant f(x_0)$. This completes the proof of the lemma. ▮

COROLLARY 1. *Let $K \subset \mathbb{R}^m$ be a convex body ($m \in \mathbb{N}$). Then for every $x^* \in \mathbb{R}^m \backslash K$ there exists a line ℓ passing through x^* with $K \cap \ell = [A, B]$, such that K possesses parallel supporting hyperplanes at A and B.*

Proof. We may assume that $x^* = 0$. Consider the corresponding functional $f(x)$ defined by (12). Let $x_0 \in K$ be the point where $f(x)$ attains its minimum. Then by Lemma 2 K possesses parallel supporting hyperplanes at $A := x_0$ and $B := x_0/f(x_0)$ and, evidently, these points belong to a line passing through the origin. ∎

Note that for *strictly* convex bodies K the above corollary can be found in [9].

Now we can determine the norm of the point-evaluation functional in the space P_n endowed with the $\|\cdot\|_{C(K)}$-norm.

THEOREM 1A. *Let $K \subset \mathbb{R}^m$ be a convex body ($m \in \mathbb{N}$), and let $x^* \in \mathbb{R}^m \backslash K$. Then*

$$\sup \left\{ \frac{|p(x^*)|}{\|p\|_{C(K)}} : p \in P_n,\, p \not\equiv 0 \right\} = T_n(\varrho(x^*, K)). \tag{18}$$

Proof. By Corollary 1 there exists a line l through x^* with $K \cap l = [A, B]$, so that for some $c \in \mathbb{R}^m$

$$\langle c, A \rangle \leqslant \langle c, x \rangle \leqslant \langle c, B \rangle, \qquad x \in K. \tag{19}$$

Set

$$\alpha := \frac{\|x^* - (A+B)/2\|}{\frac{1}{2}\,\|A - B\|}; \qquad \alpha > 1,$$

where $\|\cdot\|$ denotes the l_2-norm. Let us verify that $\varrho(x^*, K) = \alpha$. Evidently, x^* coincides with one of the points $A - (\alpha - 1)(B - A)/2$ or $B + (\alpha - 1)(B - A)/2$. Hence by Lemma 1, $x^* \in K_\alpha$. On the other hand, for arbitrary $\alpha_1 < \alpha$ we have $x^* \notin S_{\alpha_1}$, where S is the supporting strip for K defined by (19). Thus $\alpha = \varrho(x^*, K)$. Consider now an arbitrary $p \in P_n$ with $\|p\|_{C(K)} = 1$. Then $\tilde{p}(t) = p((A+B)/2 + t(B-A)/2)$ is a univariate polynomial ($t \in \mathbb{R}$) of degree at most n, and $|\tilde{p}(t)| \leqslant 1$ for $t \in [-1, 1]$. Hence by the univariate Chebyshev inequality (1) (using that $x^* = (A+B)/2 + \gamma\alpha(B-A)/2$ with $\gamma = 1$ or -1)

$$|p(x^*)| = |\tilde{p}(\gamma\alpha)| \leqslant T_n(\alpha) = T_n(\varrho(x^*, K)). \tag{20}$$

This establishes the upper bound in (18).

Set now

$$\beta := \frac{2}{\langle c, B - A \rangle} \qquad (\beta > 0)$$

and

$$p_n^*(x) := T_n\left(\beta\left\langle c, x - \frac{A+B}{2}\right\rangle\right) \in P_n. \tag{21}$$

By (19) for every $x \in K$

$$-\frac{1}{\beta} = \left\langle c, \frac{A-B}{2}\right\rangle \leqslant \left\langle c, x - \frac{A+B}{2}\right\rangle \leqslant \left\langle c, \frac{B-A}{2}\right\rangle = \frac{1}{\beta}.$$

Therefore

$$\left|\beta\left\langle c, x - \frac{A+B}{2}\right\rangle\right| \leqslant 1, \qquad x \in K,$$

i.e., for polynomial (21) we have

$$|p_n^*(x)| \leqslant 1, \qquad x \in K. \tag{22}$$

On the other hand, since $x^* = (A+B)/2 + \gamma\alpha(B-A)/2$ ($\gamma = 1$ or -1) we have by (21)

$$|p_n^*(x^*)| = \left|T_n\left(\gamma\beta\alpha\left\langle c, \frac{B-A}{2}\right\rangle\right)\right| = T_n(\alpha) = T_n(\varrho(x^*, K)).$$

This equality together with (22) yields the lower bound of Theorem 1A. ∎

Note that Theorem 1A coincides with Theorem 1 if $F = \{x^*\}$ is a singleton. Moreover, since we clearly have that

$$\varrho(F, K) = \sup_{x \in F} \varrho(x, K)$$

Theorem 1 follows directly from (18).

Remark 1. When the convex body K is, in addition, symmetric, its supporting strips can be described by (3) with $a = -b$. This fact and Lemma 1 yield that $K_\alpha = \alpha K$ and $\varrho(x, K)$ is the Minkowski functional in the symmetric case.

Remark 2. It is easy to see that both Theorems 1 and 1A fail to hold in general if K is not convex. Indeed, in this case there exists $\tilde{x} \in \text{conv}(K)\backslash K$, where $\text{conv}(K)$ is the *convex hull* of K. On the other hand, for every $\alpha > 1$ we have $K_\alpha \supset \text{conv}(K)$ because K_α is convex and $K_\alpha \supset K (\alpha > 1)$. This means that $\tilde{x} \in K_\alpha$ for any $\alpha > 1$, i.e., $\varrho(\tilde{x}, K) = 1$. Thus if, say, (18) were true for a nonconvex K then it would imply that for every $p \in P_n$ such that $|p| \leqslant 1$ on K we must also have $|p| \leqslant 1$ on $\text{conv}(K)$. But this is clearly false, in general.

MULTIVARIATE REMEZ INEQUALITY

In this section we shall study the multivariate Remez problem. This problem consists in estimating $\|p\|_{C(K)}$ for $p \in P_n$ and $K \subset \mathbb{R}^m$ provided that $|p| \leqslant 1$ on some subset $F \subset K$ satisfying $\eta_m(F) \geqslant (1-\varepsilon)\,\eta_m(K)$.

Let us introduce the corresponding quantity which provides some means of investigating the above problem: given a set $K \subset \mathbb{R}^m$ with $\eta_m(K) > 0$, set

$$\Phi_{n,m}(K,\varepsilon) = \sup\left\{\frac{\|p\|_{C(K)}}{\|p\|_{C(F)}} : p \in P_n,\ p \not\equiv 0;\ F \subset K,\ \eta_m(F) \geqslant (1-\varepsilon)\,\eta_m(K)\right\}$$

$$0 < \varepsilon < 1.$$

One of the results given in [2] states that whenever K is a convex body in $\mathbb{R}^m$

$$\Phi_{n,m}(K,\varepsilon) \leqslant T_n\left(\frac{1+\varepsilon^{1/m}}{1-\varepsilon^{1/m}}\right). \tag{23}$$

The proof of (23) given in [2] is not complete, because Lemma 1 on p. 350 is not verified in full detail (in fact, filling in the missing details there might be a nontrivial matter). Nevertheless, this problem could be avoided if instead of Lemma 2 of [2] one uses the full Remez Inequality (this makes Lemma 1 in [2] superfluous). For the sake of completeness we shall give below a proposition which is slightly more general than (23) even though its proof follows by standard arguments. Recall that given, a set $K \subset \mathbb{R}^m$ and a point $x \in K$, the set K is called *star-like* with respect to x if for any line $l \subset \mathbb{R}^m$ passing through x the set $l \cap K$ is a line segment.

PROPOSITION 1. *Let $0 < \varepsilon < 1$; $m, n \in \mathbb{N}$. Assume that $K \subset \mathbb{R}^m$ is a compact set which is star-like with respect to $x^* \in K$. Then*

$$\sup\left\{\frac{|p(x^*)|}{\|p\|_{C(F)}} : p \in P_n,\ p \not\equiv 0;\ F \subset K,\ \eta_m(F) \geqslant (1-\varepsilon)\,\eta_m(K)\right\} \leqslant T_n\left(\frac{1+\varepsilon^{1/m}}{1-\varepsilon^{1/m}}\right).$$

Proof. We may assume that $x^* = 0$ and K is star-like with respect to the origin. For $x = (x_1, ..., x_m) \in \mathbb{R}^m$ its spherical coordinates are given by

$$x_j = r \sin \varphi_{j-1} \prod_{k=j}^{m-1} \cos \varphi_k$$

$$(1 \leqslant j \leqslant m,\ r \geqslant 0;\ 0 \leqslant \varphi_1 \leqslant 2\pi;\ |\varphi_j| \leqslant \pi/2,\ 2 \leqslant j \leqslant m-1). \tag{24}$$

(Here and in what follows we assume that $\sin \varphi_0 = 1$.)

It is known that the Jacobian of this transformation can be written as

$$J = r^{m-1}\Phi(\varphi_2, ..., \varphi_{m-1}), \tag{25}$$

where Φ (shown in the proof of Lemma 3 below) is continuous on $T^{m-1} := [0, 2\pi] \times [-\pi/2, \pi/2]^{m-2}$ and $\Phi > 0$ in the interior of T^{m-1} (see [6, p. 166]). Let now $F \subset K$ be such that $\eta_m(F) \geqslant (1-\varepsilon)\, \eta_m(K)$, and denote by χ_F the characteristic function of F. Furthermore, for any fixed $\varphi \in T^{m-1}$ let $R(\varphi)$ be the ray emanating from the origin defined by (24). Since K is star-like with respect to the origin it follows that $K \cap R(\varphi)$ is a line segment. For a given $\varphi \in T^{m-1}$ let us denote by $r_1(\varphi)$ the maximal value of r for which $x = (x_1, ..., x_m)$ given by its spherical coordinates (24) is in K. Then K has the following representation in spherical coordinates:

$$K_0 := \{(\varphi, r) : \varphi \in T^{m-1};\, 0 \leqslant r \leqslant r_1(\varphi)\}. \tag{26}$$

Moreover, set

$$r_2(\varphi) = \eta_1(F \cap R(\varphi)), \qquad \varphi \in T^{m-1}. \tag{27}$$

Then by (25), (26), and the Fubini Theorem

$$\begin{aligned} \eta_m(K) &= \int_K 1\, dx = \int_{K_0} r^{m-1}\Phi(\varphi)\, d\varphi\, dr \\ &= \int_{T^{m-1}} \Phi(\varphi) \left(\int_0^{r_1(\varphi)} r^{m-1}\, dr \right) d\varphi = \frac{1}{m} \int_{T^{m-1}} \Phi(\varphi)\, r_1^m(\varphi)\, d\varphi; \end{aligned} \tag{28}$$

$$\eta_m(F) = \int_K \chi_F\, dx = \int_{T^{m-1}} \Phi(\varphi) \left(\int_0^{r_1(\varphi)} \chi_F r^{m-1}\, dr \right) d\varphi. \tag{29}$$

Using (27) we obviously have for every $\varphi \in T^{m-1}$

$$\int_0^{r_1(\varphi)} \chi_F r^{m-1}\, dr \leqslant \int_{r_1(\varphi)-r_2(\varphi)}^{r_1(\varphi)} r^{m-1}\, dr = \frac{r_1^m(\varphi) - (r_1(\varphi) - r_2(\varphi))^m}{m}. \tag{30}$$

Recalling that $\eta_m(F) \geqslant (1-\varepsilon)\, \eta_m(K)$ we have by (28)–(30)

$$(1-\varepsilon) \int_{T^{m-1}} \Phi(\varphi)\, r_1^m(\varphi)\, d\varphi \leqslant \int_{T^{m-1}} \Phi(\varphi)(r_1^m(\varphi) - (r_1(\varphi) - r_2(\varphi))^m)\, d\varphi.$$

This yields that for some $\tilde{\varphi} \in T^{m-1}$

$$(1-\varepsilon)\, r_1^m(\tilde{\varphi}) \leqslant r_1^m(\tilde{\varphi}) - (r_1(\tilde{\varphi}) - r_2(\tilde{\varphi}))^m,$$

i.e.,

$$r_2(\tilde{\varphi}) \geqslant (1-\varepsilon^{1/m})\, r_1(\tilde{\varphi}). \tag{31}$$

Note that by (24) the restriction of any $p \in P_n$ to the ray $R(\tilde{\varphi})$ is a univariate polynomial $\tilde{p}$ in the variable r of degree at most n. Hence

$$\frac{|p(0)|}{\|p\|_{C(F)}} \leqslant \frac{|\tilde{p}(0)|}{\|\tilde{p}\|_{C(F \cap R(\tilde{\varphi}))}}$$

where by (27) and (31)

$$\eta_1(F \cap R(\tilde{\varphi})) \geqslant (1-\varepsilon^{1/m})\, r_1(\tilde{\varphi}) = (1-\varepsilon^{1/m})\, \eta_1(K \cap R(\tilde{\varphi})).$$

Thus applying the univariate Remez inequality (2) transformed to the interval $K \cap R(\tilde{\varphi})$ of length $r_1(\tilde{\varphi})$ (and with $\varepsilon^{1/m}$ replacing ε) yields the statement of Proposition 1. ∎

Since a convex body is star-like with respect to any of its points the upper bound (23) follows immediately from Proposition 1.

Let us consider now the question of sharpness of (23). In [3] it is shown that equality in (23) holds if and only if K is a conic section. (A conic section is a bounded intersection of a convex cone with a half-space in $\mathbb{R}^m$.) Assuming that $0 < \varepsilon \leqslant 2^{-m}$ and using a well-known expression

$$T_n(x) = \tfrac{1}{2}\{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\}, \qquad |x| > 1,$$

we obtain from (23) that

$$\log \Phi_{n,m}(K, \varepsilon) \leqslant 6n\varepsilon^{1/2m}$$

where $K \subset \mathbb{R}^m$ is a convex body.

It can be easily shown that the above upper bound (with a different constant) remains valid for much more general domains K.

For $x \in K$ denote by $t(x)$ the maximal volume of a conic section with vertex at x which is contained in K. Similar to the proof of Proposition 1, it can be shown that

$$\log \Phi_{n,m}(K, \varepsilon) \leqslant cn\varepsilon^{1/2m}, \tag{32}$$

provided that $\inf\{t(x) : x \in K\} > 0$. This latter condition essentially means that the boundary of K does not contain cusps (that is, points which are not vertices of inscribed conic sections).

We shall consider now the following question: Can the upper bound (32) be improved (asymptotically) for *smooth* bodies K in $\mathbb{R}^m$? It turns out that $\varepsilon^{1/2m}$ in (32) can be replaced by a significantly smaller term $\varepsilon^{1/(m+1)}$ under a very mild smoothness assumption on K. Let us introduce the corresponding property. For $K \subset \mathbb{R}^m$ and $x_0 \in K$ let $r_K(x_0)$ be the radius of the largest ball contained in K such that x_0 is on the *surface* of this ball. Moreover, set

$$r(K) = \inf_{x_0 \in K} r_K(x_0).$$

We shall say that $K \subset \mathbb{R}^m$ is *smooth* if $r(K) > 0$.

Essentially, this condition requires that K has a C^2-boundary. In particular, it holds for strictly convex sets. If $K \subset \mathbb{R}^m$ is smooth then estimate (32) can be replaced by a sharper bound

$$\log \Phi_{n,m}(K, \varepsilon) \leqslant cn\varepsilon^{1/(m+1)}. \tag{33}$$

The proof of (33) differs from the method of proof of Proposition 1 in a rather significant way. Namely, in Proposition 1 we "linearized" the problem by looking at the *rays* emanating from the given point. In order to obtain the sharper bound (33) we shall use a different linearization technique based on circles passing through the given point. This will allow us to apply a univariate trigonometric version of the Remez inequality proved by Erdélyi [4]: if t_n is a univariate trigonometric polynomial of degree at most n, and $\eta_1\{\varphi \in [-\pi, \pi] : |t_n(\varphi)| \leqslant 1\} \geqslant 2\pi - \varepsilon$ $(0 < \varepsilon < \pi/2)$, then

$$\max_{|\varphi| \leqslant \pi} |t_n(\varphi)| \leqslant \exp\{c_0 n\varepsilon\}, \tag{34}$$

where $c_0 > 0$ is an absolute constant.

THEOREM 2. *Let $K \subset \mathbb{R}^m$ be smooth $(m \geqslant 2)$. Then there exist $c_1, c_2 > 0$ depending only on K and m such that*

$$\log \Phi_{n,m}(K, \varepsilon) \leqslant c_2 n\varepsilon^{1/(m+1)} \qquad (0 < \varepsilon < c_1).$$

Moreover, whenever $K \subset \mathbb{R}^m$ is compact and $\eta_m(K) > 0$ we have with some $c_3 > 0$

$$\log \Phi_{n,m}(K, \varepsilon) \geqslant c_3 n\varepsilon^{1/(m+1)} \qquad (0 < \varepsilon < c_1).$$

The proof of the above statement requires several technical auxiliary results.

LEMMA 3. *Let $m \geqslant 2$. For the transformation*

$$x_j = t \sin \varphi_{j-1} \prod_{k=j}^{m-1} \cos \varphi_k \qquad (1 \leqslant j \leqslant m-1)$$
$$x_m = t(\sin \varphi_{m-1} + 1) \tag{35}$$

where $0 \leqslant t \leqslant r$; $0 \leqslant \varphi_1 \leqslant 2\pi$; $|\varphi_j| \leqslant \pi/2$ $(2 \leqslant j \leqslant m-1)$, *the Jacobian is given by*

$$J = t^{m-1}(\sin \varphi_{m-1} + 1) \cos^{m-2} \varphi_{m-1} \tilde{\Phi}, \tag{36}$$

where $\tilde{\Phi} = \prod_{j=2}^{m-2} \cos^{j-1} \varphi_j$.

Proof. First let us examine the Jacobian J_s of the spherical coordinate transformation

$$x_j = t \sin \varphi_{j-1} \prod_{k=j}^{m-1} \cos \varphi_k \qquad (1 \leqslant j \leqslant m),$$

where $t, \varphi_1, ..., \varphi_{m-1}$ carry the same restrictions as above. Elementary properties of determinants yield

$$J_s = t^{m-1} \cos^{m-2} \varphi_{m-1} \tilde{\Phi} D_m$$

where $D_m = \det(a_{ij})$ with $a_{i1} = \sin \varphi_{i-1} \prod_{k=i}^{n-1} \cos \varphi_k$ $(1 \leqslant i \leqslant m)$ and

$$a_{ij} = \begin{cases} -\sin \varphi_{i-1} \prod_{k=i}^{j-2} \cos \varphi_k \sin \varphi_{j-1} & (1 \leqslant i < j) \\ \cos \varphi_{j-1}, & j = i \\ 0, & j > i; \end{cases}$$

$$(j = 2, 3, ..., m).$$

Now expansion of D_m along the bottom row and elementary properties of determinants reveal the recursion $D_m = D_{m-1}$. Evidently, $D_2 = 1$, and thus $D_m = 1$ for all $m \geqslant 2$. Now J differs from J_s only in that the $(m, 1)$-entry is $\sin \varphi_{m-1} + 1$ rather than $\sin \varphi_{m-1}$. Decomposing J into a sum of two determinants (where the first of them equals J_s), expanding the second determinant along the first column, and using elementary properties of determinants, we obtain

$$J = J_s + t^{m-1} \sin \varphi_{m-1} \cos^{m-2} \varphi_{m-1} \tilde{\Phi} D_{m-1}.$$

Since $D_m = \cdots = D_2 = 1$ this yields (36). ∎

LEMMA 4. *Let* $\alpha \geqslant 0$ *and* $B \subseteq [-\pi/2, \pi/2]$ *where* $\eta_1(B) = \beta < \pi/2$. *Then*

$$\int_B \cos^\alpha x(1+\sin x)\,dx \geqslant c_0 \beta^{\alpha+3}, \tag{37}$$

where $c_0 = 2^{-(\alpha+5)}\pi^{-(\alpha+2)}$.

Proof. We first argue that

$$\int_B f(x)\,dx \geqslant \int_{-\pi/2}^{-\pi/2+\beta/2} f(x)\,dx,$$

where $f(x) := \cos^\alpha x(1+\sin x)$. Let

$$B_1 := B \cap \left[-\frac{\pi}{2}, -\frac{\pi}{2}+\frac{\beta}{2}\right]; \qquad B_3 := B \cap \left[\frac{\pi}{2}-\frac{\beta}{2}, \frac{\pi}{2}\right];$$

$$B_2 := B \setminus (B_1 \cup B_3); \qquad \delta := \frac{\beta}{2} - \eta_1(B_1).$$

Then

$$\beta = \eta_1(B_1) + \eta_1(B_2) + \eta_1(B_3) \leqslant \frac{\beta}{2} - \delta + \eta_1(B_2) + \frac{\beta}{2},$$

and so $\eta_1(B_2) \geqslant \delta$. Examination of f' reveals that f is increasing on $[-\pi/2, 0]$, and it is evident that $f(-x) \leqslant f(x)$ for $0 \leqslant x \leqslant \pi/2$. As such,

$$f(y) \leqslant f\left(-\frac{\pi}{2}+\frac{\beta}{2}\right) \leqslant f(x), \qquad x \in B_2;\ y \in \left[-\frac{\pi}{2}, -\frac{\pi}{2}+\frac{\beta}{2}\right] \setminus B.$$

Thus

$$\int_B f(x)\,dx = \sum_{j=1}^{3} \int_{B_j} f(x)\,dx \geqslant \int_{B_1} f(x)\,dx + f\left(-\frac{\pi}{2}+\frac{\beta}{2}\right)\delta$$
$$\geqslant \int_{-\pi/2}^{-\pi/2+\beta/2} f(x)\,dx.$$

To complete the proof, we use the lower estimate $\sin x > (2/\pi)\,x$ for $0 < x < \pi/2$ and

$$\int_{-\pi/2}^{-\pi/2+\beta/2} f(x)\,dx \geqslant \int_{-\pi/2+\beta/4}^{-\pi/2+\beta/2} f(x)\,dx$$
$$\geqslant \frac{\beta}{4} f\left(-\frac{\pi}{2}+\frac{\beta}{4}\right) \geqslant c_0 \beta^{\alpha+3}.$$

Proof of Theorem 2. Let $F \subset K$ satisfy $\eta_m(F) \geqslant (1-\varepsilon)\,\eta_m(K)$ $(0<\varepsilon<1)$, and consider a $p \in P_n$ such that $|p| \leqslant 1$ on F. In order to verify the upper estimate of Theorem 2 we need to obtain the proper upper bound for $\|p\|_{C(K)}$. Assume that $\|p\|_{C(K)} = |p(x)|$, $x \in K$. Since K is smooth it follows that there exists a ball B_r contained in K of radius $r \geqslant r(K)$ so that x is on the surface of this ball. Furthermore, setting $A = B_r \cap (K \backslash F)$, we have $\eta_m(A) \leqslant \eta_m(K \backslash F) \leqslant \varepsilon \eta_m(K)$, and $|p| \leqslant 1$ on $B_r \backslash A$.

We may assume without loss of generality that $x=0$ and B_r is the ball with center at $(0, 0, ..., r)$ and radius $r (r \geqslant r(K))$. Consider the mapping $[0, r] \times [0, 2\pi] \times [-\pi/2, \pi/2]^{m-2} \to B_r$ given by (35) $(x = (x_1, ..., x_m) \in B_r)$. Denote by χ_A the function on B_r which equals 1 if $x \in A$, and 0 otherwise. Then using (36) we have

$$\begin{aligned}\eta_m(K)\,\varepsilon \geqslant \eta_m(A) &= \int_{B_r} \chi_A \, dx \\ &= \int_0^r \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \chi_A t^{m-1} (\sin \varphi_{m-1} + 1) \\ &\quad \times \cos^{m-2} \varphi_{m-1} \tilde{\Phi} \, d\varphi_{m-1} \cdots d\varphi_1 \, dt. \end{aligned} \tag{38}$$

For given $\varphi_{m-2}, ..., \varphi_1, t$ denote

$$\beta := \beta(\varphi_{m-2}, ..., \varphi_1, t) = \eta_1 \left\{ \varphi_{m-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] : \chi_A = 1 \right\}. \tag{39}$$

Then estimating (37) with $\alpha = m-2$ yields

$$\eta_m(K)\,\varepsilon \geqslant \int_0^r \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} c_0 \beta^{m+1} t^{m-1} \tilde{\Phi} \, d\varphi_{m-2} \cdots d\varphi_1 \, dt.$$

Since $\tilde{\Phi}$ is independent of φ_1 and is even with respect to $\varphi_2, ..., \varphi_{m-2}$ (see (36)) the last estimate easily implies that

$$\begin{aligned}\eta_m(K)\varepsilon &\geqslant c_0 \int_0^r \int_0^{\pi} \int_0^{\pi/2} \cdots \int_0^{\pi/2} \{ \beta^{m+1}(\varphi_{m-2}, ..., \varphi_1, t) \\ &\quad + \beta^{m+1}(-\varphi_{m-2}, ..., -\varphi_2, \varphi_1 + \pi, t) \} \, t^{m-1} \tilde{\Phi} d\varphi_{m-2} \cdots d\varphi_1 \, dt \\ &\geqslant c_0 \{ \beta^{m+1}(\tilde{\varphi}_{m-2}, ..., \tilde{\varphi}_1, \tilde{t}) \\ &\quad + \beta^{m+1}(-\tilde{\varphi}_{m-2}, ..., -\tilde{\varphi}_2, \tilde{\varphi}_1 + \pi, \tilde{t}) \} \, \frac{r^m}{m} C^*,\end{aligned}$$

where $(\tilde{\varphi}_{m-2}, ..., \tilde{\varphi}_1, \tilde{t})$ is some point in $[0, \pi/2]^{m-3} \times [0, \pi] \times [0, r]$, and $C^* > 0$ is a constant depending only on m (which is a result of integrating $\tilde{\Phi}$ over the corresponding domain). Recalling that $r \geqslant r(K)$, we obtain from the last estimate that

$$\beta^{m+1}(\tilde{\varphi}_{m-2}, ..., \tilde{\varphi}_1, \tilde{t}) + \beta^{m+1}(-\tilde{\varphi}_{m-2}, ..., -\tilde{\varphi}_2, \tilde{\varphi}_1 + \pi, \tilde{t}) \leqslant c_1 \varepsilon, \tag{40}$$

where $c_1 > 0$ depends only on K and m.

It is clear that the mapping (35) satisfies the property

$$x_j(\varphi_{m-1}, -\varphi_{m-2}, ..., -\varphi_2, \varphi_1 + \pi, t)$$
$$= x_j(\pi - \varphi_{m-1}, \varphi_{m-2}, ..., \varphi_1, t) \qquad (1 \leqslant j \leqslant m).$$

This and (39) yield that

$$\beta(-\tilde{\varphi}_{m-2}, ..., -\tilde{\varphi}_2, \tilde{\varphi}_1 + \pi, \tilde{t})$$
$$= \eta_1 \left\{ \varphi_{m-1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] : \chi_A(\varphi_{m-1}, -\tilde{\varphi}_{m-2}, ..., -\tilde{\varphi}_2, \tilde{\varphi}_1 + \pi, \tilde{t}) = 1 \right\}$$
$$= \eta_1 \left\{ \varphi_{m-1} \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] : \chi_A(\varphi_{m-1}, \tilde{\varphi}_{m-2}, ..., \tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{t}) = 1 \right\}.$$

Hence we obtain by (40) that for the given $(\tilde{\varphi}_{m-2}, ..., \tilde{\varphi}_1, \tilde{t})$ in $[0, \pi/2]^{m-3} \times [0, \pi] \times (0, r]$,

$$\tilde{\beta} := \eta_1 \{ \varphi_{m-1} \in [0, 2\pi] : \chi_A = 1 \} \leqslant 2(c_1 \varepsilon)^{1/(m+1)} = c_2 \varepsilon^{1/(m+1)}. \tag{41}$$

Consider now the univariate trigonometric polynomial

$$t(\varphi_{m-1}) = p(x_1, ..., x_m), \qquad \varphi_{m-1} \in [0, 2\pi]$$

where $x_1, ..., x_m$ are given by (35) with given $(\tilde{\varphi}_{m-2}, ..., \tilde{\varphi}_1, \tilde{t}) \in [0, \pi/2]^{m-3} \times [0, \pi] \times [0, r]$. Evidently, $\deg t \leqslant n$. Furthermore, by (41),

$$\eta_1 \{ \varphi_{m-1} \in [0, , 2\pi] : |t(\varphi_{m-1})| \leqslant 1 \} \geqslant 2\pi - c_2 \varepsilon^{1/(m+1)}.$$

Thus we obtain by (34) that

$$\max_{\varphi_{m-1} \in [0, 2\pi]} |t(\varphi_{m-1})| \leqslant \exp\{ c_2 n \varepsilon^{1/(m+1)} \}, \tag{42}$$

if ε is sufficiently small. Obviously, for $\varphi_{m-1}=3\pi/2$ we have by (35)

$$\left|t\left(\frac{3\pi}{2}\right)\right|=|p(0)|=\|p\|_{C(K)}.$$

This together with (42) yields the upper bound in Theorem 2.

Let us verify now the lower bound of Theorem 2.

Let B be the smallest ball in $\mathbb{R}^m$ containing K. Then $K\cap\partial B\neq\varnothing$ (∂B is the boundary of B). We may assume that $0\in K\cap\partial B$, and B is the ball with radius r and center $x_0=(r, 0, ..., 0)$. Set $F_1=\{(x_1, ..., x_m)\in B: x_1\geqslant\delta^2\}$, where $\delta=\varepsilon^{1/(m+1)}$; $F=K\cap F_1$. It is easy to see that $B\backslash F_1$ is contained in an m-dimensional rectangular solid with sides $h_1=\delta^2$; $h_2=\cdots=h_m=2\sqrt{2r}\,\delta$. Thus $\eta_m(B\backslash F_1)\leqslant c_0\,\delta^{m+1}=c_0\varepsilon$. Therefore

$$\eta_m(K\backslash F_1)\leqslant\eta_m(B\backslash F_1)\leqslant c_0\varepsilon,$$

and hence

$$\begin{aligned}\eta_m(F)&=\eta_m(K)-\eta_m(K\backslash F_1)\\&\geqslant\eta_m(K)-c_0\varepsilon=\eta_m(K)(1-c\varepsilon),\end{aligned}\tag{43}$$

where $c>0$ depends only on K.

Consider now the polynomial

$$p_n(x):=T_n\left(\frac{2x_1-2r-\delta^2}{2r-\delta^2}\right),\qquad x=(x_1, ..., x_m)\in\mathbb{R}^m.$$

Since $\delta^2\leqslant x_1\leqslant 2r$ whenever $x\in F_1$ it follows that $|p_n(x)|\leqslant 1$ for $x\in F\subset F_1$. On the other hand, $0\in K$ and

$$\begin{aligned}|p_n(0)|&=\left|T_n\left(\frac{2r+\delta^2}{2r-\delta^2}\right)\right|\geqslant\left|T_n\left(1+\frac{\delta^2}{r}\right)\right|\geqslant\exp(c^*n\delta)\\&=\exp(c^*n\varepsilon^{1/(m+1)}).\end{aligned}$$

This together with (43) yields the lower bound in Theorem 2. ∎

Summarizing the discussion of the multivariate Remez problem we can conclude that for domains K without cusps we have $\log\Phi_{n,m}(K,\varepsilon)=O(n\varepsilon^{1/2m})$, while for smooth domains K a sharper bound $\log\Phi_{n,m}(K,\varepsilon)=O(n\varepsilon^{1/(m+1)})$ holds. This improvement in the ε-term can make a difference when one applies these estimates in order to prove Nikolski-type inequalities. (Such applications first appeared in the multivariate setting in [2].) Nikolski-type inequalities provide estimates from above for uniform norms of polynomials via their L_p-norm. Let ω be positive η_m a.e. on $K\subset\mathbb{R}^m$ and set $\|g\|_{L_p(K)}=(\int_K \omega\,|g|^p)^{1/p}$.

Consider the quantity

$$\beta_{n,m}(\omega, K)=\sup\left\{\frac{\|g\|_{C(K)}}{\|g\|_{L_p(K)}}: g\in P_n,\ g\not\equiv 0\right\}.$$

In order to estimate $\beta_{n,m}(\omega, K)$ we shall use the function

$$\varphi(\omega, \delta)=\inf\left\{\int_A \omega: A\subset K,\ \eta_m(A)\geqslant\delta\right\}.$$

Assume that for the domain K we have a Remez-type estimate

$$\log \Phi_{n,m}(K, \varepsilon)\leqslant cn\varepsilon^{\gamma}$$

with some $0<\gamma<1$. (As we have seen above $\gamma=1/2m$ for domains without cusps, and $\gamma=1/(m+1)$ for smooth domains.) Denote by $\delta_n(\omega)$ the unique solution of the equation

$$\varphi(\omega, \delta)=e^{-n\delta^{\gamma}}. \tag{44}$$

Then it can be shown that

$$\log \beta_{n,m}(\omega, K)\leqslant cn\delta_n^{\gamma}(\omega). \tag{45}$$

The proof of this inequality can be obtained similarly to the proof of Lemma 2 in [7], where the univariate version of (45) is given (based on the univariate Remez inequality (34).) It is clear that the magnitude of γ can have a direct effect on estimate (45). For instance, if the weight ω is such that $\varphi(\omega, \delta)\sim e^{-1/\delta}$, then $\delta_n(\omega)\sim n^{-1/(1+\gamma)}$, and in view of (44) we obtain

$$\log \beta_{n,m}(\omega, K)\leqslant cn^{1/(1+\gamma)}. \tag{46}$$

Hence, in particular, when $\gamma=1/2m$ (domains without cusps) the corresponding estimate in (46) is $n^{2m/(1+2m)}$, while for $\gamma=1/(m+1)$ (smooth domains) we get $n^{(m+1)/(m+2)}$. This observation illustrates the surprising fact that the magnitude of the quantity $\beta_{n,m}(\omega, K)$ appearing in the Nikolski-type inequalities in the multivariate case may depend on the smoothness of the boundary of the underlying domain.

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